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Published in:
Annals of Pure and Applied Logic

DOI:
10.1016/0168-0072(92)90035-X

Annals of Pure and Applied Logic 58 (1992) 73–92
North-Holland

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Communicated by A. Prestel
Received 5 April 1991

Abstract

Van den Berg, I.P., Extended use of IST, Annals of Pure and Applied Logic 58 (1992) 73–92.

Internal Set Theory is an axiomatic approach to nonstandard analysis, consisting of three axiom schemes, Transfer (T), Idealization (I), and Standardization (S). We show that the range of application of these axiom schemes may be enlarged with respect to the original formulation. Not only more kinds of formulas are allowed, but also different settings. Many examples illustrate these extensions. Most concern formal aspects of nonstandard asymptotics.

1. Introduction

1.1. On Internal Set Theory

The original approach to nonstandard analysis by A. Robinson [17, 1973] was based on the extension of the 'standard' model of analysis $\mathbb{R}$ to a larger 'nonstandard' model $^*\mathbb{R}$. The latter contains, next to a copy of $\mathbb{R}$, nonstandard elements, in particular infinitesimals and infinitely large numbers.

The extension chosen by Robinson is an ultrapower, hence $\mathbb{R}$ and $^*\mathbb{R}$ are elementary equivalent. The immediate consequence of this is that every theorem concerning real numbers, formulated in the usual formal set theory ZFC, can be proved in a new way, i.e. by proving the corresponding theorem within $^*\mathbb{R}$. Thus Robinson concluded that nonstandard analysis, more than creating new mathematical entities, consists of introducing new deductive procedures.

Such new principles of deduction were stated formally by E. Nelson in 1977 [13]. He introduced a new unary predicate symbol 'st' (for 'standard') and three axiom schemes. Formulas which do not contain the symbol 'st' are called internal, and formulas which do contain the symbol 'st' are called external. The first axiom

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scheme is the principle of transfer:

(T) $\quad (\forall^{st} t)((\forall^{st} x)\, A(x, t) \Leftrightarrow (\forall x)\, A(x, t)).$

Here A is internal. The second axiom scheme is the principle of idealization:

(I) $\quad (\forall^{stfin} z)(\exists y)(\forall x \in z)\, B(x, y) \Leftrightarrow (\exists y)(\forall^{st} x)\, B(x, y).$

Here B is some internal formula, which may contain free variables. The third axiom scheme is the principle of standardization:

(S) $\quad (\forall^{st} X)(\exists^{st} y)(\forall^{st} z)(z \in y \Leftrightarrow z \in X \wedge C(z)).$

The formula C may be internal or external and may also contain free variables. N.B. $(\forall^{st} x)$ is an abbreviation of $(\forall x)(\text{st}x \rightarrow \cdots)$, $(\exists^{st} x)$ is an abbreviation of $(\exists x)(\text{st}x \wedge \cdots)$ and $(\forall^{stfin} x)$ is an abbreviation of $(\forall x)(\text{st}x \wedge x \text{ finite} \Rightarrow \cdots)$. Together with the axioms of ZFC the above axioms form Internal Set Theory (IST). The Internal Set Theory resumes for a large part the practice of Robinsonian nonstandard analysis, but from a theoretical point of view there is an important difference: nonstandard objects exist already within standard sets. Thus the infinitesimals are part of $\mathbb{R}$ instead of some extension of $\mathbb{R}$. Introductions to IST are included in [8], [9], [12], [13], [15] or [16]; for notions and notations concerning elementary nonstandard analysis, we refer to [8] or [12].

The structure of IST is at the same time simple and powerful. Up to equivalence, there are only three types of closed external formulas: formulas of the form $(\exists^{st} x)\, A(x)$, of the form $(\forall^{st} x)\, A(x)$, or of the form $(\forall^{st} x)(\exists^{st} y)\, A(x, y)$ where A is always supposed to be internal. The formulas of the first type, called *galactic,* and the formulas of the second type, called *halic,* are incompatible, i.e., a formula which is both equivalent to a galactic and a halic formula must be internal. The 'nonstandard world' and the 'standard world' are connected in a straightforward manner by the Reduction Algorithm, which transforms every external theorem and proof into an internal theorem and proof. Finally, IST is completely saturated. The above properties were shown by Nelson in [13] and [14], except for the incompatibility result proved in [1].

The Internal Set Theory has been adopted by quite a number of working mathematicians, and this number is growing. It appears to be accessible to non-logicians, and to be an efficient tool in dealing with practical and theoretical problems. Important contributions have been made notably in the domain of asymptotic analysis in the wider sense (singular perturbations, divergent series, ordinary differential equations), but also in probability theory, the study of the moiré-technique, the desingularization of algebraic manifolds and the classification of Lie-algebra's. Bibliographies may be found in [7], [9], [12] and [19].

1.2. Aim of this paper

In this paper we show that within IST the principles of transfer, idealization, standardization and saturation may be generalized, or be applied in entirely new situations. The idealization principle may be generalized from internal to halic formulas. The principles of saturation and standardization, whose original formulation concerns only the standard elements of a set, may under some restrictions also be applied to the nonstandard elements of a set. Some generalizations of the transfer principle, such as the uniqueness principle, had already been remarked by Nelson. Here we prove two transfer principles of a different kind: a 'function criterion', related to the sequence criterion for continuity of ordinary analysis, and the so-called monadic transfer principle. Essentially, the latter says that every property expressed within IST (without using nonstandard constants) can be transferred from a given internal set I to its monad, i.e. the intersection of all the standard sets containing I.

Many examples illustrate the new principles. Most of them concern formal properties of standard and nonstandard asymptotic analysis. For instance, generalized idealization yields a short proof of the existence theorem relative to the nonstandard shadow expansions. Generalized standardization readily turns such a nonstandard existence theorem into a standard existence theorem relative to the classical asymptotic expansions. The function criterion shows the equivalence of certain notions within the theory of regular variation. A particular case of the monadic transfer principle constitutes an important step in the proof of a general theorem on the asymptotic behaviour of solutions of differential equations.

The use of this logical machinery within asymptotics may surprise somewhat, but it has been preceded by the 'Cauchy principle' and the 'Fehrele principle'. The first is based on the incompatibility of internal and external formulas, and the second on the incompatibility of galactic and halic formulas, mentioned above. These frequently used principles have been very successful in dealing with, usually delicate, matching problems.

1.3. Conventions and notation

Strictly speaking IST only concerns internal sets, i.e. sets defined by internal formulas, and does not dispose of 'external sets'. However, like most of the mathematicians working within IST we adopt the use of external sets as far as they have only internal elements, and are 'defined' by a formula of IST. In this context statements involving external sets may be considered as abbreviations of formulas. External sets induce a certain neatness and flexibility into argumentation and description. For instance they are very adapted to describe accurately qualitative or asymptotic behaviour of functions.

Often it is convenient to define external sets by the union or intersection of internal families of sets, rather than by external formulas. The following notions will be used throughout this paper.

A union of the form

$$G = \bigcup_{\mathrm{st}x \in X} A_x$$

where X is standard and $(A_x)_{x\in X}$ is an internal family of sets is called a *pregalaxy*; if it is external G is called a *galaxy*. An intersection of the form

$$H = \bigcap_{\mathrm{st}x\in X} A_x$$

where X is standard and $(A_x)_{x\in X}$ is an internal family of sets is called a *prehalo*, if it is external H is called a *halo*. Thus a (pre)galaxy is defined by a galactic formula and a (pre)halo is defined by a halic formula. Note that external sets can always be reduced to the forms

$$\bigcup_{\mathrm{st}x\in X}\ \bigcap_{\mathrm{st}y\in Y} A_{xy},$$

where X and Y are standard, and $(A_{xy})_{x\in X, y\in Y}$ is an internal family of sets.

Sometimes we wish to precise the index set X in the definitions of (pre)galaxy and (pre)halo. Then we speak about *X-galaxies, X-halos* etc. For instance, the set of real infinitesimals

$$\bigcap_{\mathrm{st}n\in\mathbb{N}} \left[\frac{1}{n}, \frac{1}{n}\right]$$

is an $\mathbb{N}$-halo.

A formula of IST is called *absolute* if it does not contain nonstandard parameters. For instance if $f:\mathbb{R}\to\mathbb{R}$ is standard, and $\varepsilon\in\mathbb{R}$, then $(\forall x \simeq +\infty)(f(x)\simeq 0)$ is absolute and $(\forall x\simeq +\infty)(f(x)\leq \varepsilon)$ is not absolute. A (pre)galaxy is called *absolute* if it is of the form $\bigcup_{\mathrm{st}x\in X} A_x$, where st$X$ and $(A_x)_{x\in X}$ is a standard family of sets. Note that a (pre)galaxy or (pre)halo is absolute if and only if it is defined by an absolute formula.

We recall some notation concerning particular external sets. The set of real infinitesimals will be written $\emptyset$, the set of positive real infinitesimals will be written $\emptyset^+$, the set of limited real numbers will be written £ and the set of positive appreciable (i.e. limited, but not infinitesimal) numbers will be written @. We may 'calculate' with these symbols in an obvious way ($\emptyset\cdot\emptyset=\emptyset$, $\emptyset\cdot £ = \emptyset$, $@\cdot £ = £$, $1/(1+\emptyset) = 1+\emptyset$ etc.). In some cases the equality sign should be interpreted as an inclusion, which is not uncommon within asymptotics. For instance $f(\omega) = (1+\emptyset)\omega$ is another way of saying that $f(\omega)/\omega \simeq 1$. Sometimes the set of real numbers infinitely close to a given real number r will be written hal(r). The set of positive limited real numbers will be written hal$(+\infty)$.

Finally an internal or external set will be called *purely nonstandard* if all its elements are nonstandard.

1.4. Structure of this paper

Chapter 2 is devoted to the idealization principle and its consequences. The idealization principle will be generalized from internal formulas to halic formulas. Applications include simple proofs of the existence theorem for shadow-expansions in various settings.

In Chapter 3 we present an extension of the saturation principle and of the standardization principle in its modified form S'. These extensions are similar and concern the existence of an internal 'choice function' under external conditions. We also consider situations where such a choice function does not exist. Among the applications are simple proofs of Du Bois–Reymond's lemma and the Borel–Ritt existence theorem for asymptotic expansions.

Chapter 4 is devoted to extensions of the transfer principle. We first recall some direct generalizations, already observed by Nelson. Section 4.1 contains a function criterion; a special case corresponds to the sequence criterion for the continuity of functions within ordinary analysis. In Section 4.2 we first define the monad of a set. We then state the very general monadic transfer principle, which transfers all absolute properties from a given prehalo to its monad. Special care is given to the problem how to determine the monad of a set. We give some applications, which suggest that the function criterion and the monadic transfer principle are useful in two-parameter problems and in the problem of how to transform local asymptotic behaviour into global asymptotic behaviour. The applications concern notably the theory of regular variation and the asymptotics of differential equations.

2. Generalized idealization

The idealization principle is crucial to nonstandard analysis, for it generates nonstandard elements in any infinite set. Other important consequences are the principles of extension and saturation, the Fehrele principle and a compactness and finite intersection property.

For a brief account of extension and saturation we refer to the next section. The Fehrele principle states that no halo is a galaxy. This incompatibility yields permanence results: halic properties verified on galaxies still hold somewhere beyond, and vice-versa. A high amount of literature testifies to the usefulness of such results in nonstandard asymptotics (see, among others, [2] or [19]). The Fehrele principle is a direct consequence of a separation theorem: for every galaxy G and halo H such that $G \subset H$ there exists an internal set I such that $G \subset I \subset H$. A short proof is the following. Let st$S$, T, let $(A_s)_{s\in S}$ be an internal family such that $G = \bigcup_{\mathrm{st}s\in S} A_s$ and let $(B_t)_{t\in T}$ be an internal family such that $H = \bigcap_{\mathrm{st}t\in T} B_t$. For every standard finite $u \subset S$, $v \subset T$ there exists an internal set I such that $A_s \subset I \subset B_t$ for all $s \in u$, $t \in v$; take, say, $I = \bigcup_{\mathrm{st}s\in u} A_s$. By idealization there exists an internal set I such that $A_s \subset I \subset B_t$ for all standard $s \in S$, $t \in T$. Hence $G \subset I \subset H$.

The compactness property and the finite intersection property will be used throughout this paper, and run as follows.

Compactness property. Let H be a prehalo, let T be standard, and let $(A_t)_{t\in T}$ be an internal family of (internal) sets. Assume that $H \subset \bigcup_{\mathrm{st}\, t \in T} A_t$. Then there is a standard finite set $z \subset T$ such that $H \subset \bigcup_{\mathrm{st}\, t \in z} A_t$.

Finite intersection property. Let G be a pregalaxy, let T be standard and let $(A_t)_{t \in T}$ be an internal family of (internal) sets. Assume that $\bigcap_{\mathrm{st}\, t \in T} A_t \subset G$. Then there is a standard finite set $z \subset T$ such that $\bigcap_{\mathrm{st}\, t \in z} A_t \subset G$.

The finite intersection property is just the complementary form of the compactness property. The latter is easily derived from the separation principle and the contraposition of (I):

$$(\forall y)(\exists^{\mathrm{st}} x)\, C(x, y) \Leftrightarrow (\exists^{\mathrm{stfin}} z)(\forall y)(\exists x \in z)\, C(x, y). \tag{2.1}$$

Again $C(x, y)$ is an internal property. To obtain the compactness property, notice that H and $G \equiv \bigcup_{\mathrm{st}\, t \in T} A_t$ are separated by some internal set I, and then apply (2.1) with $C(t, y)\colon y \in I \Rightarrow y \in A_t$.

The main result of this chapter is the following generalization of the idealization principle.

Theorem 2.1 (Halic idealization principle). *Let $H(x, y)$ be a halic property. Then*

$$(\forall^{\mathrm{stfin}} z)(\exists y)(\forall x \in z)\, H(x, y) \Leftrightarrow (\exists y)(\forall^{\mathrm{st}} x)\, H(x, y). \tag{2.2}$$

Proof. The formula $H(x, y)$ is of the form $(\forall^{\mathrm{st}} s)\, A(s, x, y)$ where A is an internal property. By interchanging quantifiers and applying the usual idealization axiom we see that the left-hand side of (2.2) is equivalent to

$$(\forall^{\mathrm{stfin}} z)(\forall^{\mathrm{stfin}} w)(\exists y)(\forall x \in z)(\forall s \in w)\, A(s, x, y). \tag{2.3}$$

So $A(s, x, y)$ holds on every cartesian product of standard finite sets $z \times w$. This is equivalent to saying that $A(s, x, y)$ holds on every standard finite set u of couples (s, x). Indeed, on the one hand u is included in the cartesian product of its projections and on the other hand every cartesian product of standard finite sets is itself standard finite. Hence (2.3) is equivalent to

$$(\forall^{\mathrm{stfin}} u)(\exists y)(\forall c \equiv (s, x) \in u)\, A(s, x, y). \tag{2.4}$$

By the usual idealization axiom (2.4) is equivalent to

$$(\exists y)(\forall^{\mathrm{st}} c = (s, x))\, A(s, x, y). \tag{2.5}$$

Finally (2.5) is equivalent to the right member of (2.2). $\square$

The contraposition of (2.2) is valid for galactic formulas $G(x, y)$ and reads

$$(\forall y)(\exists^{\mathrm{st}} x)\, G(x, y) \Leftrightarrow (\exists^{\mathrm{stfin}} z)(\forall y)(\exists x \in z)\, G(x, y). \tag{2.6}$$

Example (*Existence theorems for shadow expansions*). The nonstandard notion of shadow expansion of numbers corresponds to the classical notion of asymptotic expansion of functions. For an introduction to the shadow expansions, notation and terminology we refer to [2] or [8].

A question of theoretical interest is the following. Let the sequence of positive real numbers $(u_k)_{k\in\mathbb{N}}$ be an order scale, i.e. such that $u_{k+1}/u_k \simeq 0$ at least for st$k \in \mathbb{N}$, and let $(c_k)_{k\in\mathbb{N}}$ be a standard sequence of real numbers. Does the formal series $\sum_{k=0}^{\infty} c_k u_k$ have a 'sum', i.e., does there exist a real number s such that the series is the shadow expansion of s with respect to the given order scale? The answer is affirmative, and we even have the following precision: there exists an (unlimited) integer ν such that the partial sums $\sum_{k=0}^{\mu} c_k u_k$ have the prescribed expansion for all $\mu \leq \nu$, $\mu \simeq +\infty$. Several proofs have been given; they have in common that ν depends on the sequence $(c_k)_{k\in\mathbb{N}}$, but in an external way.

However, it appeared that there exists a 'universal' index ρ, valid for all standard sequences. Using halic idealization we may give a simple proof of this fact.

Let $n \in \mathbb{N}$, and let $s_n \equiv \sum_{k=0}^{n} c_k u_k$ be the nth partial sum of the series $\sum_{k=0}^{\infty} c_k u_k$. As usual, we write $s \sim \sum_{k=0}^{\infty} c_k u_k$ if $\sum_{k=0}^{\infty} c_k u_k$ is the shadow expansion of the number s.

Proposition 2.2. *Let $(u_k)_{k\in\mathbb{N}}$ be an order scale. Then there exists an unlimited index ρ such that $s_\rho \sim \sum_{k=0}^{\infty} c_k u_k$ for all standard sequences of real numbers $(c_k)_{k\in\mathbb{N}}$.*

Proof. It follows readily from its definition that the notion "$s \sim \sum_{k=0}^{\infty} c_k u_k$" is halic. For every standard sequence $(c_k)_{k\in\mathbb{N}}$ there exists $\nu \simeq +\infty$ such that $s_\mu \sim \sum_{k=0}^{\infty} c_k u_k$ for all $\mu \simeq +\infty$, $\mu \leq \nu$, so certainly there exists an unlimited index ρ common to any standard finite set of sequences. By halic idealization there exists $\rho \simeq +\infty$ such that $s_\rho \sim \sum_{k=0}^{\infty} c_k u_k$ for all standard sequences. □

Shadow expansions of functions have also been considered. They consist of formal series $\sum_{k=0}^{\infty} f_k u_k$, where the f_k form a standard sequence of real functions; it follows from the 'theorem of the continuous shadow' that the functions are necessarily continuous (see [8, p. 89]). Shadow expansions of functions are useful in nonstandard perturbation theory (see for instance [7]). Halic idealization yields a quick proof for the following existence theorem; as usual we denote the nth partial sum by s_n.

Proposition 2.3. *Let $(u_k)_{k\in\mathbb{N}}$ be an order scale and $(f_k)_{k\in\mathbb{N}}$ be a standard sequence of real (continuous) functions. Then there exists an unlimited index ν such that*

$$s_\mu \sim \sum_{k=0}^{\infty} f_k u_k \quad \textit{for all } \mu \simeq +\infty,\ \mu \leq \nu.$$

Proof. To apply idealization, it is sufficient to show that for every st$m \in \mathbb{N}$ and st$a, b \in \mathbb{R}$ there exists an index $n > m$ such that

$$(\forall \mu, m \leqslant \mu \leqslant n)(\forall x \in [a, b])((s_\mu(x) - s_m(x))/u_m \simeq 0). \tag{2.7}$$

We may take $n > m$ to be any standard index. Then, indeed for μ such that $m \leqslant \mu \leqslant n$,

$$(s_\mu(x) - s_m(x))/u_m = \sum_{k=m+1}^{\mu} f_k(x) u_k / u_m \simeq 0$$

because the $f_k(x)$ are limited and the quotients u_k/u_m are infinitesimal. By halic idealization, there exists $\nu \simeq +\infty$ such that (2.7) holds for all st$m$, and for all limited x. Hence, by the definition of shadow expansions of real functions, $s_\mu \sim \sum_{k=0}^{\infty} f_k u_k$ for all $\mu \simeq +\infty$, $\mu \leqslant \nu$. $\square$

There exists also a universal index, valid for all standard sequences of continuous real functions; this may be proved in a similar way as Proposition 2.2.

From Proposition 2.2 we will derive an existence theorem for classical asymptotic expansions in the next chapter, using extended standardization.

3. Extended standardization and saturation

Both standardization and saturation may be formulated in a way close to the axiom of choice, and it is in such a form that they will be extended.

It has been shown by Nelson in [13] and [14] that the standardization principle may be modified as follows:

(S′) $\quad (\forall^{\mathrm{st}} x)(\exists^{\mathrm{st}} y)\, \Phi(x, y) \Leftrightarrow (\exists^{\mathrm{st}} \tilde{y})(\forall^{\mathrm{st}} x)\, \Phi(x, \tilde{y}(x)).$

The formula Φ is an arbitrary formula of IST. It is tacitly supposed that the x range over a standard set X, the y range over a standard set Y, and $\tilde{y}$ is a standard mapping from X to Y.

It has also been shown by Nelson [14] that the following form of the saturation principle is a theorem of IST:

$$(\forall^{\mathrm{st}} x)(\exists y)\, \Phi(x, y) \Leftrightarrow (\exists \tilde{y})(\forall^{\mathrm{st}} x)\, \Phi(x, \tilde{y}(x)).$$

Again the formula Φ is an arbitrary formula of IST and it is tacitly supposed that the x range over a standard set X; the difference with (S′) lies in the fact that the y are allowed to be internal nonstandard, and thus the mapping $\tilde{y}$ may also be internal nonstandard.

Often the modified standardization principle and the saturation principle are used as an extension principle: every external function defined only on the standard elements of a standard set X, with standard (respectively internal) values, may be extended to a standard (respectively internal) function defined on the whole of X.

The next theorems show that under some restrictions standardization or saturation is possible also on the nonstandard elements of a set.

Theorem 3.1 (Standardization on absolute $\mathbb{N}$-halos). *Let* stX, Y *and* $H \subset X$ *be a purely nonstandard absolute* $\mathbb{N}$*-halo. Let* $\Phi(x, y)$ *be an absolute galactic or* $\mathbb{N}$*-halic property such that for all* $x \in H$ *there exist* $y \in Y$ *such that* $\Phi(x, y)$. *Then there exists a standard mapping* $\tilde{y}: X \to Y$ *such that* $\Phi(x, \tilde{y}(x))$ *for all* $x \in H$.

Proof. Put $E = \{(x, y) \in X \times Y \mid \Phi(x, y)\}$. We distinguish two cases: (i) E is a pregalaxy; (ii) E is an $\mathbb{N}$-halo.

(i) *E is a pregalaxy.* Let st$T$ and $(A_t)_{t \in T}$ be a standard family of subsets of $X \times Y$ such that $E = \bigcup_{\mathrm{st} t \in T} A_t$. Then

$$P_X(E) = \bigcup_{\mathrm{st} t \in T} P_X(A_t) \supset H.$$

By compactness there is a standard finite z such that $C \equiv \bigcup_{\mathrm{st} t \in z} P_X(A_t) \supset H$. Let $B = \bigcup_{\mathrm{st} t \in z} A_t$. Then B and C are standard, $P_X(B) = C$ and $B \subset E$. By modified standardization, and transfer, there is a standard mapping $\tilde{y}: C \to Y$ such that $(x, \tilde{y}(x)) \in B$ for all $x \in C$. Then certainly $(x, \tilde{y}(x)) \in E$ for all $x \in H$. The domain of $\tilde{y}$ may trivially be extended to X.

(ii) *E is an* $\mathbb{N}$*-halo.* Without restriction of generality we may assume that $P_X(E) = H$. Let $(B_n)_{n \in \mathbb{N}}$ be a standard strictly decreasing sequence of subsets of $X \times Y$ such that $B_0 = X \times Y$ and $\bigcap_{\mathrm{st} n \in \mathbb{N}} B_n = E$. Put $C_n = P_X(B_n)$ for every $n \in \mathbb{N}$. Then $(C_n)_{n \in \mathbb{N}}$ is a standard decreasing sequence of subsets of X such that $H = \bigcap_{\mathrm{st} n \in \mathbb{N}} C_n$. Again without restriction of generality we may assume that $(C_n)_{n \in \mathbb{N}}$ is strictly decreasing because H is external. Then by modified standardization, and transfer, for every st$n \in \mathbb{N}$ there exists a standard mapping $\tilde{y}_n: C_n \backslash C_{n+1} \to Y$ such that $(x, \tilde{y}_n(x)) \in B_n$ for every $x \in C_n \backslash C_{n+1}$. Let $\tilde{y}: X \to Y$ be the standardized of $\bigcup_{\mathrm{st} n \in \mathbb{N}} \tilde{y}_n$. Let $x \in H$. Because H is purely nonstandard, we have $\bigcap_{n \in \mathbb{N}} C_n = \emptyset$. So there exists $\omega \simeq +\infty$ such that $x \in C_\omega \backslash C_{\omega+1}$. Then $(x, \tilde{y}(x)) = (x, \tilde{y}_\omega(x)) \in B_\omega \subset E$. □

Theorem 3.2 (Saturation on halos). *Let* stX, Y *and* $H \subset X$ *be a non-empty prehalo. Let* $\Phi(x, y)$ *be a galactic or* $\mathbb{N}$*-halic property such that for all* $x \in H$ *there exists* $y \in Y$ *such that* $\Phi(x, y)$. *Then there exists an internal mapping* $\tilde{y}: X \to Y$ *such that* $\Phi(x, \tilde{y}(x))$ *for all* $x \in H$.

Proof. Put $E = \{(x, y) \in X \times Y \mid \Phi(x, y)\}$. If E is a pregalaxy, the proof is similar to part (i) of the proof of Theorem 3.1. So let us assume that E is a halo.

Let $(B_n)_{n \in \mathbb{N}}$ be an internal decreasing sequence of (internal) subsets of $X \times Y$ such that $E = \bigcap_{\mathrm{st} n \in \mathbb{N}} B_n$, we may assume that $B_0 = X \times Y$. For every $n \in \mathbb{N}$ we put $C_n = P_X(B_n)$. Then $(C_n)_{n\in\mathbb{N}}$ is also decreasing. Put $K = \bigcap_{\mathrm{st}n\in\mathbb{N}} C_n$. Then $K \supset H$. If K is internal, then $K = C_\nu$ for some $\nu \simeq +\infty$. By the axiom of choice there is an internal function $\tilde{y}: C_\nu \to Y$ such that $(x, \tilde{y}(x)) \in B_\nu$ for every $x \in K$. Then certainly $(x, \tilde{y}(x)) \in E$ for every $x \in H$. If K is external, we may assume without loss of generality that $(C_n)_{n\in\mathbb{N}}$ is strictly decreasing for $\mathrm{st}n \in \mathbb{N}$. By the Cauchy principle there is $\omega \in \mathbb{N}$, $\omega \simeq +\infty$ such that (i) $(C_n)_{n\leqslant\omega}$ is strictly decreasing and (ii) $C_\omega \neq \emptyset$. By the axiom of choice for every $n < \omega$ there exists an internal function $\tilde{y}_n: C_n \backslash C_{n+1} \to Y$ such that $(x, \tilde{y}_n(x)) \in B_n$ for every $x \in C_n \backslash C_{n+1}$, and a function $\tilde{y}_\omega: C_\omega \to Y$ such that $(x, \tilde{y}_\omega(x)) \in B_\omega$ for every $x \in C_\omega$. Let $\tilde{y} = \bigcup_{n\leqslant\omega} \tilde{y}_n$. Then $(x, \tilde{y}(x)) \in E$ for every $x \in K$, hence certainly for every $x \in H$. □

We give three applications of standardization on halo's. The first and second concern nonstandard proofs of existence theorems of asymptotics, and the third concerns a useful particular case.

Proposition 3.3 (Du Bois–Reymond's lemma). *Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of real strictly positive functions such that for every $n \in \mathbb{N}$ one has $f_{n+1}(x) = o(f_n(x))$ for $x \to +\infty$. Then there is a real strictly positive function f such that for every $n \in \mathbb{N}$ one has $f(x) = o(f_n(x))$ for $x \to +\infty$.*

Proof. By transfer, we may assume that the sequence is standard. Let $\omega \simeq +\infty$, and $\mathrm{st}n \in \mathbb{N}$. Then $f_n(\omega) = \emptyset \cdot f_{n-1}(\omega) = \cdots = \emptyset \cdot f_0(\omega)$. It follows from Robinson's lemma (or from halic idealization) that there exists $\mu > 0$ such that $\mu = \emptyset \cdot f_n(\omega)$ for all $\mathrm{st}n \in \mathbb{N}$. By halic standardization there exists a standard strictly positive real function f such that $f(\omega) = \emptyset \cdot f_n(\omega)$ for all $\omega \simeq +\infty$ and $\mathrm{st}n \in \mathbb{N}$. Then for every $\mathrm{st}n \in \mathbb{N}$ one has that $f(x) = o(f_n(x))$ for $x \to +\infty$. By transfer, the same holds for all $n \in \mathbb{N}$. □

Proposition 3.4 (Existence theorem for asymptotic expansions; Borel–Ritt theorem). *Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of strictly positive real functions such that for every $n \in \mathbb{N}$ one has $f_{n+1}(x) = o(f_n(x))$ for $x \to +\infty$. Let $(a_n)_{n\in\mathbb{N}}$ be a sequence of real numbers. Then there is a real function s such that the formal series $\sum_{n=0}^{\infty} a_n f_n$ is the asymptotic expansion of s.*

Proof. By transfer, we may assume that the sequences $(f_n)_{n\in\mathbb{N}}$ and $(a_n)_{n\in\mathbb{N}}$ are standard. Let $\omega \simeq +\infty$. Then the sequence $(f_n(\omega))_{n\in\mathbb{N}}$ is an order scale. By Proposition 2.2 there is a real number σ such that $\sigma \sim \sum_{n=0}^{\infty} a_n f_n(\omega)$. By halic standardization there is a standard real function s such that $s(\omega) \sim \sum_{n=0}^{\infty} a_n f_n(\omega)$ for every $\omega \simeq +\infty$. Then $\sum_{n=0}^{\infty} a_n f_n$ is an asymptotic expansion of f for $x \to +\infty$. □

Lemma 3.5. *Let* stX, Y. *Let $H \subset X$ be a purely nonstandard absolute $\mathbb{N}$-halo and let $K \subset Y$ be an absolute $\mathbb{N}$-prehalo such that $K \neq \emptyset$. Then there is a standard function $f: X \to Y$ such that $f(H) \subset K$.*

Proof. Define $L = H \times K$. By Theorem 3.1 there is a standard function $f: X \to Y$ such that $f_{|H} \subset L$. This means that $f(H) \subset K$. □

The next generalization and its corollary will be used in Chapter 4.

Convention. By 'halic and/or $\mathbb{N}$-galactic' properties we mean properties which are internal, or of the form $(\forall^{st}x)\, A(x)$, $(\exists^{st}n \in \mathbb{N})\, A(n)$, $(\forall^{st}x)(\exists^{st}n \in \mathbb{N})\, A(x, n)$ or $(\exists^{st}n \in \mathbb{N})(\forall^{st}x)\, A(x, n)$, where A is internal. A 'galactic and/or $\mathbb{N}$-halic' property is just the negation of a halic and/or $\mathbb{N}$-galactic property.

Lemma 3.6. *Let* stX, Y. *Let $H \subset X$ be a purely nonstandard absolute $\mathbb{N}$-halo and let $K \subset Y$ be non-empty and defined by an absolute galactic and/or $\mathbb{N}$-halic formula. Then there exists a standard function $f: X \to Y$ such that $f(H) \subset K$.*

Proof. We distinguish two cases:

$$\text{(i) } K = \bigcup_{\text{st}t \in T} \bigcap_{\text{st}n \in \mathbb{N}} A_{nt}, \qquad \text{(ii) } K = \bigcap_{\text{st}n \in \mathbb{N}} \bigcup_{\text{st}t \in T} A_{nt},$$

where st$T$ and $(A_{nt})_{n \in \mathbb{N}, t \in T}$ is a standard family of subsets of Y.

(i) $K = \bigcup_{\text{st}t \in T} \bigcap_{\text{st}n \in \mathbb{N}} A_{nt}$. We write $M_t = \bigcap_{\text{st}n \in T} A_{nt}$ for every $t \in T$. For some st$t \in T$ we have $M_t \neq \emptyset$. By Lemma 3.5 there exists a standard function $f: X \to Y$ such that $f(H) \subset M_t$. Then certainly $f(H) \subset E$.

(ii) $K = \bigcap_{\text{st}n \in \mathbb{N}} \bigcup_{\text{st}t \in T} A_{nt}$. We write $G_n = \bigcup_{\text{st}t \in T} A_{nt}$. Now $G_n \neq \emptyset$ for all st$n \in \mathbb{N}$. So for every st$n \in \mathbb{N}$ there exists st$t \in T$ such that $A_{nt} \neq \emptyset$. By standardization there exists a standard sequence $(t_n)_{n \in \mathbb{N}}$ such that $A_{nt_n} \neq \emptyset$ for all st$n \in \mathbb{N}$. Put $F = \bigcap_{\text{st}n \in \mathbb{N}} A_{nt_n}$. By Lemma 3.5 there exists a standard function $f: X \to Y$ such that $f(H) \subset F$. Then certainly $F(H) \subset K$. □

Corollary 3.7. *Let* st$X$ *and $K \subset X$ be non-empty, and defined by an absolute galactic, and/or $\mathbb{N}$-halic property. Then there exists a standard sequence $(x_n)_{n \in \mathbb{N}}$ such that $x_n \in K$ for all $n \simeq +\infty$.*

It is interesting to note some situations where standardization or saturation on halo's does not work, i.e., where there does not exist a standard or internal 'choice function'.

(1) Let $H = \{x \in \mathbb{R} \mid x \simeq 0\}$ and $E = \{(x, y) \in \mathbb{R}^2 \mid x \simeq 0, y > 0\}$. There is no standard function $\varphi: \mathbb{R} \to \mathbb{R}$ such that $\varphi_{|H} \subset E$ for $(0, \varphi(0))$ should be standard, and E does not contain standard points. This does not contradict the theorem on standardization on halo's, for H is not purely nonstandard. Of course, there exists an internal choice function.

(2) Let $H = \{x \in \mathbb{R} \mid x \simeq +\infty\}$ and $\mathcal{F} = \{f : \mathbb{R} \to \mathbb{R} \mid \lim_{x \to +\infty} f(x) = +\infty\}$. Put

$$E = \{(x, y) \in \mathbb{R} \mid x \simeq +\infty,\ y \simeq +\infty,\ (\forall^{\mathrm{st}} f \in \mathcal{F})(y < f(x))\}.$$

The E is a halo, but it easily follows from Du Bois–Reymond's lemma that E is not an $\mathbb{N}$-halo. By the Fehrele principle, for all $x \simeq +\infty$ there exists $y \in \mathbb{R}$ such that $(x, y) \in E$, but clearly there does not exist a standard function φ such that $(x, \varphi(x)) \in E$ for all $x \simeq +\infty$. As it has been shown in [2], there does not exist an internal function with such properties either, so even saturation does not hold. This means that there does not exist an internal function which 'separates' the bounded standard functions from the standard functions going off to positive infinity, on the whole halo of infinity.

(3) Let $H = \{x \in \mathbb{Q} \mid \mathrm{nst} x\}$ and let

$$E = \{(x, y) \in \mathbb{Q}^2 \mid x \underset{\not\simeq}{<} 0,\ y \underset{\simeq}{\leq} 0 \text{ or } x \underset{\simeq}{\geq} 0,\ y \underset{\not\simeq}{>} 0\}.$$

Then H is a purely nonstandard absolute $\mathbb{N}$-halo and E is neither a halo nor a galaxy. Clearly the vertical projection of E contains H. But there does not exist a standard or internal choice-function, for it is a direct consequence of what was proved in [2, p. 116] that every internal function $f : \mathbb{Q} \to \mathbb{Q}$ necessarily touches E^{c}.

4. Generalized transfer

By the transfer principle an internal absolute property is transferred from the (external) set of all standard elements of a given set X to the set X itself. If X is infinite, the transfer is non-trivial.

The subject of this section is to study more situations where a property may be transferred from a given set to a bigger set. First we give a short account of direct generalizations of (T). In Section 4.1 we present a function criterion, related to the sequence criterion for continuity of functions of ordinary analysis. Finally in Section 4.2 we prove the monadic transfer principle which transfers absolute properties from a particular set to the intersection of all standard sets containing this set.

The following direct generalizations of (T) were already considered by Nelson.

(i) The transfer principle (T) holds also for absolute halic formulas.

(ii) The contraposition of (T)

$$(\exists x)\, A(x) \Leftrightarrow (\exists^{\mathrm{st}} x)\, A(x), \tag{4.1}$$

also holds for absolute galactic formulas.

(iii) (Uniqueness principle). Let A be an internal absolute formula. Then it follows immediately from (4.1) that

$$\exists! x\, A(x) \Leftrightarrow \mathrm{st} x. \tag{4.2}$$

This uniqueness principle is used to prove that the explicitly defined objects of classical analysis, such as 1, 2, π, $\mathbb{N}$, $\mathbb{R}$ are standard.

For proofs of the above results we refer to [13] or [14].

4.1. A function criterion

We recall that 'halic and/or $\mathbb{N}$-galactic' properties are properties which are internal, or of the form $(\forall^{st}x)\,A(x)$, $(\exists^{st}n\in\mathbb{N})\,A(n)$, $(\forall^{st}x)(\exists^{st}n\in\mathbb{N})\,A(x,n)$ or $(\exists^{st}n\in\mathbb{N})(\forall^{st}x)\,A(x,n)$ where A is internal.

Theorem 4.1 (External function criterion). *Let* stX,Y *and let* $H\subset X$ *be a purely nonstandard absolute* $\mathbb{N}$*-halo and* $K\subset Y$ *be a non-empty absolute* $\mathbb{N}$*-prehalo.*

(1) (Transfer from images). *Let* $\Phi(y)$ *be an absolute halic and/or* $\mathbb{N}$*-galactic property. Assume* Φ *holds on* $f(H)$ *for every standard mapping* $f:X\to Y$ *such that* $f(H)\subset K$. *Then* Φ *holds on* K.

(2) (Transfer from graphs). *Let* $\Phi\subset X\times Y$ *be an absolute halic and/or* $\mathbb{N}$*-galactic property. Assume* Φ *holds on* $f_{|H}$ *for every standard mapping* $f:X\to Y$ *such that* $f(H)\subset K$. *Then* Φ *holds on* $H\times K$.

Comment. If there exists a standard function f such that $f(H)\supset K$, the first part of the above theorem is of course trivial. However, it is non-trivial in, say, the following case. Let $(B_n)_{n\in\mathbb{N}}$ be a standard decreasing sequence of subsets of X such that $H=\bigcap_{\mathrm{st}n\in\mathbb{N}}B_n$ and $(C_n)_{n\in\mathbb{N}}$ be a standard decreasing sequence such that $K=\bigcap_{\mathrm{st}n\in\mathbb{N}}C_n$. Assume the B_n all have infinite cardinality B and the C_n all have cardinality C, and that $B<C$. Let us further assume that $B_0=X$. We show that

$$\bigcup\{f(H)\mid \mathrm{st}f:X\to Y, f(H)\subset K\}\subsetneq K.$$

Indeed, let $f:X\to Y$ be an arbitrary standard mapping. Then clearly $f(X)\not\supset K$, i.e., there exists $y\in K$ not lying in the image of f. The same holds for the union of the image of a (standard) finite set of mappings. By halic idealization there is $y\in K$ not in the image of any standard mapping $f:X\to Y$ and in particular those mappings such that $f(H)\subset K$.

The transfer from graphs is nontrivial whenever K is external. Indeed, by the Fehrele principle for every $x\in H$ the galaxy $\{f(x)\mid \mathrm{st}f,\ f(H)\subset K\}$ is strictly included in the halo K.

Note that the theorem cannot be extended to arbitrary galactic properties. As a counterexample to part (1) take $\Phi(y)=(\exists^{st}f:\mathbb{N}\to\mathbb{R})(y\in f(\mathbb{N}))$, $H=\{n\in\mathbb{N}\mid n\simeq+\infty\}$ and $K=\mathrm{hal}(0)$, and as a counterexample to part (2) take $\Phi(x,y)=(\exists^{st}f:\mathbb{R}\to\mathbb{R})(y=f(x))$, $H=\mathrm{hal}(\infty)$, and $K=\mathrm{hal}(0)$.

Proof of Theorem 4.1. (1) Put $E=\{y\in Y\mid \Phi(y)\}$. Suppose $E^c\cap K\neq\emptyset$. By Lemma 3.6 there exists a standard mapping $f:X\to Y$ such that $f(H)\subset E^c\cap K$. So Φ does not hold on $f(H)$, a contradiction. We conclude that $K\subset E$, i.e., Φ holds on K.

(2) Put $E=\{(x,y)\in X\times Y \mid \Phi(x,y)\}$ and $L=E^{c}\cap H\times K$. Suppose $L\neq\emptyset$. By Corollary 3.7 there exists a standard sequence $(x_n, y_n)_{n\in\mathbb{N}}$ of elements of $X\times Y$ such that $(x_n, y_n)\in L$ for all $n\simeq+\infty$. By, if necessary, taking a subsequence we may assume that $(x_n)_{n\in\mathbb{N}}$ is injective. Let $s=\{x_n \mid n\in\mathbb{N}\}$. Define $f_1: s\to Y$ by $f_1(x_n)=y_n$ for all $n\in\mathbb{N}$. By Lemma 3.6 there exists a standard function $f_2: X\backslash s\to Y$ such that $f(H\backslash s)\subset K$. Put $f=f_1\cup f_2$. Then $f_{|H}\subset K$, but Φ does not hold on $f_{1|H}$. So we have a contradiction. Hence $L=\emptyset$, i.e., Φ holds on $H\times K$. □

As a corollary we present a sequence criterion. Then we will give some examples and applications.

Corollary 4.2 (External sequence criterion). *Let* st$X$ *and* $H\subset X$ *be a nonempty absolute* $\mathbb{N}$*-prehalo. Let* S *be the external set of all standard sequences* $(x_n)_{n\in\mathbb{N}}$ *such that* $x_\omega\in H$ *for all* $\omega\simeq+\infty$ *and* $K=\{x_\omega \mid x\in S,\ \omega\simeq+\infty\}$. *Let* $\Phi(x)$ *be a halic and/or* $\mathbb{N}$*-galactic absolute property. If* Φ *holds for every* $x\in K$, *then* Φ *holds for every* $x\in H$.

Example 1 (*Classical sequence criterion*). The well-known sequence criterion for limits of real functions

$$\lim_{x\to p} f(x)=q \Leftrightarrow \lim_{n\to+\infty} f(x_n)=q \quad \text{for all sequences } (x_n)_{n\in\mathbb{N}} \text{ such that } \lim_{n\to+\infty} x_n=p \tag{4.3}$$

is a standard counterpart of a special case of Corollary 4.2. Indeed, let $p,q\in\mathbb{R}$ and $f:\mathbb{R}\to\mathbb{R}$ be standard. Then the left member of (4.3) is equivalent to

$$(\forall x\simeq p)\, f(x)\simeq q \tag{4.4}$$

and the right member of (4.3) is equivalent to

$$(\forall (x_n)_{n\in\mathbb{N}})((\forall\omega\simeq+\infty)(x_\omega\simeq p)\Rightarrow(\forall\omega\simeq+\infty)(f(x_\omega)\simeq q)). \tag{4.5}$$

Because the property $f(x)\simeq q$ is halic, by Corollary 4.2 the formulas (4.4) and (4.5) are equivalent. Notice that the transfer (4.5)$\Rightarrow$(4.4) is nontrivial, as follows from a cardinality argument.

Example 2 (*A function criterion with additional conditions*).

Proposition 4.3. *Let* $P(x,y)$ *be an absolute halic and/or* $\mathbb{N}$*-galactic property such that* $P(x,g(x))$ *holds for every* $x\simeq+\infty$ *and every standard continuous function* $g:\mathbb{R}\to\mathbb{R}$ *such that* $g(x)\simeq+\infty$ *for all* $x\simeq+\infty$. *Then* $P(x,y)$ *holds for all* $x\simeq+\infty$, $y\simeq+\infty$.

Proof. We first show that $P(x,f(x))$ holds for every $x\simeq+\infty$, where $f:\mathbb{R}\to\mathbb{R}$ is any standard function such that $f(x)\simeq+\infty$ for all $x\simeq+\infty$. Indeed, let (x_n, y_n) be a standard sequence such that $x_n \simeq +\infty$ and $y_n = f(x_n)$ for all $n \simeq +\infty$. By choosing a subsequence, if necessary, we may assume that $(x_n)_{n\in\mathbb{N}}$ is strictly increasing. Then the sequence may be extended to a standard continuous function g such that $g(x) \simeq +\infty$ for all $x \simeq +\infty$. So the property holds for all $(x, g(x))$ such that $x \simeq +\infty$, and in particular for all (x_n, y_n) such that $n \simeq +\infty$. By Corollary 4.2 we have $P(x, f(x))$ for all $x \simeq +\infty$. Hence $P(x, y)$ for all $x \simeq +\infty$, $y \simeq +\infty$ by Theorem 4.1. □

It follows from the proof that the above proposition allows for many variations: the functions g may be supposed differentiable, C^∞ and/or monotonous, we may transfer to different products of $\mathbb{N}$-halos, like $\{(x, y) \mid x \simeq 0, x > 0, y \simeq 0, y > 0\}$, $\{(x, y) \mid x \simeq +\infty, y \simeq 0\}$ etc.

Let us give a few examples and applications. We leave the verification of the details to the reader.

(i) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be standard. Assume that f is S-continuous on every set $\{(x, g(x)) \mid x \simeq +\infty, \mathrm{st}g : \mathbb{R} \to \mathbb{R}, \lim_{t\to\infty} g(t) = +\infty\}$. Then f is S-continuous at every point (x, y) with $x \simeq +\infty$, $y \simeq +\infty$.

(ii) If $\mathrm{st}f : \mathbb{R}^2 \to \mathbb{R}$ is limited at all $x \simeq +\infty$ along every standard continuous function g such that $\lim_{x\to+\infty} g(x) = 0$, then f is standardly bounded on some box $[A, \infty) \times [-m, m]$ with st$A$ and st$m > 0$.

(iii) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be of class C^2 and G be the set of all functions $f : \mathbb{R} \to \mathbb{R}$ of class C^2 such that $g(0) = 0$. Put $\psi(y) = f(0, y)$ and $\psi_g(x) = f(x, g(x))$ for every $g \in G$. Assume that

(a) $\quad \varphi'(0) = 0, \quad \varphi''(0) < 0,$

(b) $\quad \psi_g'(0) = 0, \quad \psi_g''(0) < 0$ for every $g \in G$,

then f has a maximum in $(0, 0)$.

Two earlier papers contain results which may be considered as instances of the external function criterion. The first result [5] concerns the equivalence for standard functions of the notions of 'asymptotic continuity' and 'macroscopic observability'. A real function F is said to be *asymptotically continuous* if for all real functions G such that $G(X) \sim X$ for $X \to +\infty$ we have $F(G(X)) \sim F(X)$ for $X \to +\infty$ and F is said to be *observable by macroscope* if $\forall\omega \simeq +\infty$ the shadow of its image $f_\omega(x) \equiv F(\omega x)/F(\omega)$ under the substitution $X = \omega x$, $Y = G(\omega)y$ is a well defined (single-valued) function on at least $(0, \infty)$. Further, put $H(X, Y) = F(Y)/F(X)$. A crucial step in the proof of the equivalence of the above two notions consists in showing that $H(\omega, G(\omega)) \simeq 1$ for all $\omega \simeq +\infty$ and for all standard functions G such that $G(X) = (1 + \emptyset)X$ for all $X \simeq +\infty$ if and only if $H(\omega, Y) \simeq 1$ for all $Y = (1 + \emptyset)\omega$. Another instance of the external function criterion in [5] concerned the proof that 'asymptotically bounded' functions, — i.e. functions F which satisfy the property that $F(G(X)) = O(F(X))$ for $X \to +\infty$ for all functions G such that $G(X) \sim X$ for $X \to +\infty$ — are of polynomial growth.

Special cases of the external function criterion also occur in [4], again to prove the equivalence of standard and nonstandard notions relative to standard objects. These notions concerned firstly a sort of asymptotic continuity for functions of two variables, and secondly a mathematical model for exponential contraction, relative to the optical 'river-phenomenon' appearing in phase portraits of differential equations.

4.2. *Monadic transfer*

In this last section we first recall the notion of monad. We show how in many cases absolute properties may be transferred from a given set to its monad. In some elementary, but fairly general cases we indicate how the monad of a set may be determined. Finally we give some applications within analysis and topology.

Definition 4.1. Let st$X$ and $E \subset X$ be an internal or external set. The *monad* $M(E)$ of E is defined by

$$M(E) = \bigcap \{S \subset X \mid \text{st}S,\ E \subset S\}.$$

There are other occurrences of the terminology 'monad' within nonstandard analysis (see [20]), and the different notions should be well distinguished. The *monad of a set E relative to a standard topology* is the intersection of all standard neighbourhoods of E. The *monad of a standard family* $\mathcal{F}$ *of sets* (for instance a filter) is the intersection of all standard members of $\mathcal{F}$.

Theorem 4.4 (Monadic transfer principle). *Let* st$X$ *and let* $H \subset X$ *be a prehalo. Let* $P(x)$ *be an absolute property. If* P *holds on* H, *then* P *also holds on* $M(H)$.

Proof. Put $E = \{x \in X \mid P(x)\}$. Then

$$E = \bigcap_{\text{st}u \in U} \bigcup_{\text{st}v \in V} C_{uv}$$

where stU,V and $(C_{uv})_{u \in U, v \in V}$ is a standard family of subsets of X. For every $u \in U$, define

$$G_u = \bigcup_{\text{st}v \in V} C_{uv}.$$

Then $H \subset G_u$ for every st$u \in U$. By compactness, for every st$u \in U$ the prehalo H is covered by a standard finite number of the C_{uv}, so for every st$u \in U$ there exists a standard set A such that $H \subset A \subset G_u$. By standardization there exists a standard family $(A_u)_{u \in U}$ such that $H \subset A_u \subset G_u$ for every st$u \in U$. Then

$$M(H) \subset \bigcap_{\text{st}u \in U} A_u \subset \bigcap_{\text{st}u \in U} G_u = E.$$

Hence P holds on $M(H)$. $\square$

Corollary 4.5. *Let* st$X$ *and* $H \subset X$ *be a prehalo. Let* $P(x)$ *be an absolute property. If there exists* $x \in M(H)$ *such that* $P(x)$ *holds, then there exists* $x \in H$ *such that* $P(x)$ *holds.*

To render the monadic transfer principle operational, we must have ways to determine the monad of sets, or to recognize those sets whose monad is a given absolute halo. The next propositions are useful in this context. They will be illustrated by various examples, where we concentrate on $\mathbb{R}$ and $\mathbb{R}^2$.

Proposition 4.6. *Let* st$X$ *and* $H \subset X$ *be an absolute prehalo. Let* $G \subset H$ *be a purely nonstandard pregalaxy. Then*

$$M(H \backslash G) = H.$$

Proof. Clearly $M(H\backslash G) \subset M(H) = H$. Conversely, let V be an arbitrary standard set with $H\backslash G \subset V$. Then $H \subset V \cup G$. It follows from compactness that there exists a standard set U such that $H \subset U \subset V \cup G$. Because G has no standard elements, by transfer we have $U \subset V$. So $H \subset V$. Hence $H \subset M(H\backslash G)$, because V was arbitrary. We conclude that $H = M(H\backslash G)$. □

Examples. Let $\varepsilon, \omega \in \mathbb{R}$, where $\varepsilon \simeq 0$, $\varepsilon > 0$, and $\omega \simeq +\infty$. The following equalities are immediate consequences of Proposition 4.6.

(1) $M\{x \in \mathbb{R} \mid x \simeq +\infty, x \leqslant \omega\} = \text{hal}(+\infty)$.

(2) $M\{(x, y) \in \mathbb{R}^2 \mid x \simeq 0, y \simeq 0, |x|, |y| \geqslant \varepsilon\} = \text{hal}^2(0) - \{0\}$.

(3) $M(\mathbb{R}^2 - \{(x, y) \mid \exists^{\text{st}} s, t \in \mathbb{R}, x = \varepsilon + s \text{ or } y = \varepsilon + t\}) = \mathbb{R}^2$.

Proposition 4.7. *Let* st$X$ *and* $\mathscr{A}$ *be a standard family of subsets of* X. *Put* $H = \bigcap \{\text{st}A \mid A \in \mathscr{A}\}$, *and let* $B \in \mathscr{A}$. *Then*

$$M(H \cap B) = H.$$

Proof. If B is standard, there is nothing to prove, so let us assume that B is nonstandard. Clearly $M(H \cap B) \subset M(H) = H$. Conversely, let V be an arbitrary standard set such that $H \cap B \subset V$. By the finite intersection property there exists a standard finite set $z \subset \mathscr{A}$ such that $\bigcap \{A \mid A \in z\} \cap B \subset V$. Noting that $\bigcap \{A \mid A \in z\}$ is standard there exists st$C \in \mathscr{A}$ such that $\bigcap \{A \mid A \in z\} \cap C \subset V$, by transfer. So $H \subset V$ and hence $H \subset M(H \cap B)$. We conclude that $M(H \cap B) = H$. □

Corollary 4.8. (1) *Let* st$X$ *and* $\mathscr{A}$ *be a standard family of subsets of* X. *Put* $H = \bigcap \{\text{st}A \mid A \in \mathscr{A}\}$. *Assume that* $B \in \mathscr{A}$ *is such that* $B \subset H$. *Then* $M(B) = H$.

(2) *Let* st$X$ *and* $(A_n)_{n \in \mathbb{N}}$ *be a standard decreasing sequence of subsets of* X. *Let* $\omega \simeq +\infty$. *Then*

$$M(A_\omega) = \bigcap_{\text{st}n \in \mathbb{N}} A_n.$$

Examples. (1) Let $\omega \simeq +\infty$. Then $M([\omega, \infty[) = \mathrm{hal}(\infty)$. This is an immediate consequence of Corollary 4.8(2).

(2) Let $\omega \simeq +\infty$, $\varepsilon \simeq 0$, $\varepsilon > 0$. Then $M([\omega, \infty[\times [0, \varepsilon]) = \mathrm{hal}(\infty) \times \emptyset^+$: this follows directly if we apply Corollary 4.8(1) to the family of rectangles $\{[a, \infty) \times [0, b] \mid a, b > 0\}$.

(3) Let X be a standard topological space and $x \in X$. A neighbourhood U of x will be called *infinitesimal* if U is contained in every standard neighbourhood, i.e., $U \subset \mu(x)$ where $\mu(x)$ is the monad of x in the topological sense. Then $M(U) = \mu(x)$ by Corollary 4.8. Note that by idealization every point has an infinitesimal neighbourhood.

Finally, we give some applications. The first concerns a property which had already been observed by Robinson [17, p. 79].

Proposition 4.9. *Let $f: \mathbb{R} \to \mathbb{R}$ be standard and $\omega \simeq +\infty$. If $f(x) \simeq 0$ for all $x \simeq +\infty$, $x \leqslant \omega$, then $f(x) \simeq 0$ for all $x \simeq +\infty$ (and thus $\lim_{x \to \infty} f(x) = 0$).*

The next proposition can be reduced to Proposition 4.9.

Proposition 4.10. *Let $f, g: \mathbb{R} \to \mathbb{R}$ be standard and $\omega \simeq +\infty$. If $f(x) = (1 + \emptyset)g(x)$ for all $x \simeq +\infty$, $x \leqslant \omega$, then $f(x) = (1 + \emptyset)g(x)$ for all $x \simeq +\infty$ (and thus $f(x) \sim g(x)$ for $x \to +\infty$).*

Proposition 4.10 constituted an important step in the nonstandard proof of the existence of 'rivers' as presented in [3] and [4]. Rivers are exponentially stable or unstable standard solutions of standard differential equations, and Proposition 4.10 was notably useful in the unstable case. Indeed, using elementary geometric means it appeared to be possible to obtain a standard solution $f(x)$ following asymptotically a certain standard function $g(x)$ for all $x \simeq +\infty$ up to some $\omega \simeq +\infty$, however, due to the unstability, it was not possible a priori to obtain asymptotic closeness for all $x \simeq +\infty$. Yet Proposition 4.10 enabled us to deduce the global asymptotic behaviour from the local asymptotic behaviour.

The next application concerns topological monads.

Proposition 4.11. *Let X be a standard topological space. Let $x \in X$ and $\mu(x)$ be the monad of X with respect to the topology. Let E be a standard or an external absolute subset of X such that $E \cap \mu(x) \neq \emptyset$. Then $x \in \bar{E}$.*

Proof. Suppose $x \notin \bar{E}$. Then there exists a neighbourhood U of x such that $U \cap E = \emptyset$. So $U \subset E^c$ and also $U \cap \mu(x) \subset E^c$. Now $\mu(x) = M(U \cap \mu(x)) \subset E^c$ by Proposition 4.7 and monadic transfer. So we have a contradiction. We conclude that $x \in \bar{E}$. $\square$

We obtain the following corollary. Let X be a standard T_1-space, and $\mathrm{st}x \in X$. Then for every $y \in \mu(x)$ it holds that $x \in \partial\mu(y)$. Indeed, we have $x \in \overline{\mu(y)}$ by Proposition 4.11, and $x \notin \mu(y)$ because $X - \{x\}$ is a standard neighbourhood of y.

The final application concerns the notion of 'slow oscillations'. There are two very different formulations of this notion and we propose to prove their equivalence. The proof uses both the external function criterion and the monadic transfer principle.

Definition 4.2 (Schmidt [18], Hardy [11]). A real function f is called *slowly oscillating* if for all functions g with $g(x) \geqslant x$ for all $x \in \mathbb{R}$

$$g(x) \sim x \quad \text{for } x \to +\infty \quad \Rightarrow \quad f(g(x)) - f(x) = o(1) \quad \text{for } x \to +\infty. \tag{4.6}$$

Definition 4.3 (Bingham et al. [6]). A real function f is called *slowly oscillating* if

$$\lim_{\lambda \downarrow 1} \limsup_{x \to +\infty} \sup_{t \in [1,\lambda]} |f(tx) - f(x)| = 0. \tag{4.7}$$

By transfer, to show the equivalence of (4.6) and (4.7), we only need to consider standard f. We then prove that (4.6) and (4.7) are both equivalent to

$$(\forall \omega \simeq +\infty) f((1 + \emptyset^+)\omega) = f(\omega) + \emptyset. \tag{4.8}$$

An example of a slowly oscillating function is given by the function $\sin \log x$. This is perhaps shown in the most easy way using (4.8).

Proposition 4.12. *For standard f the formulas* (4.6) *and* (4.8) *are equivalent.*

Proof. Put

$$h(x, y) = f(y) - f(x).$$

Then (4.6) is equivalent to

$$(\forall^{\mathrm{st}} g)[(\forall \omega \simeq +\infty)(g(\omega) = (1 + \emptyset^+)\omega) \Rightarrow (\forall \omega \simeq +\infty)(h(\omega, g(\omega)) = \emptyset)]. \tag{4.9}$$

The equivalence of (4.8) and (4.9) is then a direct consequence of the external function criterion. □

Proposition 4.13. *For standard f the formulas* (4.7) *and* (4.8) *are equivalent.*

Proof. The implication (4.8)$\Rightarrow$(4.7) is straightforward. Conversely, define the standard functions φ and ψ by

$$\varphi(\lambda, x) = \sup_{t \in [1,\lambda]} |f(tx) - f(x)|, \qquad \psi(\lambda) = \limsup_{x \to +\infty} \varphi(\lambda, x).$$

Let $\lambda \simeq 1$, $\lambda \geq 1$. Then $\psi(\lambda) \simeq 0$ by the nonstandard characterization of the limit. So there exists $\xi \simeq +\infty$ such that $\varphi(\lambda, \omega) \simeq 0$ for all $\omega \geq \xi$. This means that $f(t\omega) - f(\omega) \simeq 0$ for all $t \in [1, \lambda]$ and $\omega \geq \xi$. By monadic transfer (in fact, as a consequence of example (2) above) we have that $f(t\omega) - f(\omega) \simeq 0$ for all $t \simeq 1$, $t \geq 1$ and $\omega \simeq +\infty$. Hence $f((1 + \emptyset^+)\omega) = f(\omega) + \emptyset$ for all $\omega \simeq +\infty$. $\square$

Corollary 4.14. *The characterizations* (4.6) *and* (4.7) *of slow oscillation are equivalent.*

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